

Ellipses of minimal eccentricity inscribed in midpoint diagonal quadrilaterals.

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9/2/16

Abstract

In [3] we showed that there is a unique ellipse of minimal eccentricity, E_I , inscribed in any convex quadrilateral, Q . Using a different approach than in [3], we prove that there is a unique ellipse of minimal eccentricity, E_I , inscribed in a midpoint diagonal quadrilateral, Q , which is a quadrilateral with the property that the intersection point of the diagonals of Q coincides with the midpoint of at least one of the diagonals of Q . Our main result is that if Q is a midpoint diagonal quadrilateral, then the smallest non-negative angle between equal conjugate diameters of E_I equals the smallest non-negative angle between the diagonals of Q . This was proven in [4] for parallelograms.

1 Introduction

In [3] the author proved numerous results about ellipses inscribed in convex quadrilaterals, Q . By *inscribed* we mean that the ellipse lies inside Q and is tangent to each side of Q . In particular, we proved that there exists a unique ellipse, E_I , of minimal eccentricity inscribed in Q . In [4] we gave the following geometric characterization of E_I for parallelograms: The smallest nonnegative angle, Γ , between equal conjugate diameters of E_I equals the smallest nonnegative angle, α , between the diagonals of Q . The main result in this paper (Theorem 3) is to extend this result to a larger class of quadrilaterals we call midpoint diagonal quadrilaterals. A quadrilateral, Q , is called a midpoint diagonal quadrilateral if the intersection point of the diagonals of Q coincides with the midpoint of at least one of the diagonals of Q . This includes the possibility that Q is a parallelogram, in which case the diagonals of Q bisect one another. Equivalently, if Q is not a parallelogram, then Q is a midpoint diagonal quadrilateral if and only if the line thru the midpoints of the diagonals of Q contains one of the diagonals of Q . For convex quadrilaterals in general, some simple examples show that $\Gamma \neq \alpha$. In addition, other examples show that there are convex quadrilaterals which are not midpoint diagonal quadrilaterals with $\Gamma = \alpha$.

The method of proof in this paper is somewhat different than in [3], where we used a theorem of Marden relating the foci of an ellipse tangent to the lines

thru the sides of a triangle and the zeros of a partial fraction expansion. In this paper we use formulas for the lengths of the major and minor axes of an ellipse, E_0 , inscribed in Q , as a function of the coefficients of an equation of E_0 (Lemma 12), and hence we have a formula for the eccentricity of E_0 as a function of the coefficients as well. This approach is shorter and more direct. As noted above, Theorem 3 holds for parallelograms. Also, we show below (Lemma 10) that a midpoint diagonal quadrilateral cannot be a trapezoid. Thus we may assume throughout this paper (unless stated otherwise) that no two sides of Q are parallel.

2 Locus of Centers of Ellipses inscribed in Quadrilaterals

Before proving our main result, Theorem 3, we need more general results about ellipses inscribed in quadrilaterals which will be useful later. The following problem, often referred to in the literature as Newton's problem, is to determine the locus of centers of ellipses inscribed in Q . The solution given by Newton is described in the following theorem (see [1] and [2]).

Theorem 1 (*Newton*): *Let M_1 and M_2 be the midpoints of the diagonals of a quadrilateral, Q . If E is an ellipse inscribed in Q , then the center of E must lie on the open line segment, Z , connecting M_1 and M_2 .*

Remark 1 *If Q is a parallelogram, then the diagonals of Q intersect at the midpoints of the diagonals of Q , and thus Z is really just one point.*

The classical proof of Theorem 1 involves first using an orthogonal projection to map E to a circle, C , and then proving Theorem 1 for C . Affine invariance then allows one to obtain Theorem 1 for ellipses in general. However, Theorem 1 does not really give the precise locus of centers of ellipses inscribed in Q , but only shows that the center of E must lie on Z . What about the converse? That is, is **every point** of Z the center of some ellipse inscribed in Q ? The following theorem shows that the locus of centers of ellipses inscribed in Q is precisely Z .

Theorem 2 [3] *Let Q be a convex quadrilateral in the xy plane with no two sides parallel. Let M_1 and M_2 be the midpoints of the diagonals of Q , and let Z be the open line segment connecting M_1 and M_2 . If $(h, k) \in Z$, then there is a unique ellipse with center (h, k) inscribed in Q .*

Theorem 2 was proven in [3] using a result of Marden relating the foci of an ellipse tangent to the lines thru the sides of a triangle and the zeros of a partial fraction expansion. We provide a different proof here and we also fill in a gap in [3] for the proof of uniqueness. The following proposition allows us to fill that gap.

Proposition 1 Suppose that E_1 and E_2 are distinct ellipses with the same center and which are each inscribed in the same convex quadrilateral, Q . Then Q must be a parallelogram.

Remark 2 Chakerian([1]) mentions the essence of Proposition 1, but no proof is cited or given.

Proof. Note that ellipses, tangent lines to ellipses, convex quadrilaterals, and parallelograms are preserved under nonsingular affine transformations. First, since E_1 and E_2 have the same center, by applying a translation, one can assume that E_1 and E_2 have center = $(0,0)$; Now use a scaling transformation which maps E_1 to the unit circle and thus leaves E_2 as an ellipse. Finally, by applying a rotation about the origin, we can assume that E_2 has major and minor axes parallel to the x and y axes, respectively. The equations of E_1 and E_2 are then $x^2 + y^2 = 1$ with slope function $\frac{dy}{dx} = -\frac{x}{y}, y \neq 0$, and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with slope function $\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}, y \neq 0$; Note that $a \neq b$ since $E_1 \neq E_2$ and hence $a > b$; Since E_1 and E_2 are each inscribed in the same convex quadrilateral, there must be four distinct lines which are tangent to each of the curves E_1 and E_2 ; If $a = 1$, then there are only two lines (the vertical lines $x = \pm 1$) which are tangent to both E_1 and E_2 , while if $b = 1$, then that there are only two lines (the horizontal lines $y = \pm 1$) which are tangent to both E_1 and E_2 . Thus we may assume that $a \neq 1 \neq b$; It is then clear that any line tangent to both E_1 and E_2 cannot be horizontal or vertical. So suppose that the line, L , is tangent to E_1 and E_2 at the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, respectively. Then $-1 < x_1 < 1, -1 < y_1 < 1, -a < x_2 < a$, and $-b < y_2 < b$ since L is not vertical. Also, L has equation

$$y = mx + B, m \neq 0 \quad (1)$$

since L is not horizontal. Note that x_1, y_1, x_2 , and y_2 are all nonzero. We then have $x_1^2 + y_1^2 = 1, \frac{dy}{dx} = -\frac{x_1}{y_1}$ if $y_1 \neq 0$ for P_1 and $\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1, \frac{dy}{dx} = -\frac{b^2 x_2}{a^2 y_2}$ if $y_2 \neq 0$ for P_2 ; Since L is tangent to E_1 and E_2 at P_1 and P_2 , respectively, the equation of L is also given by $y = -\frac{x_1}{y_1}x + y_1 + \frac{x_1^2}{y_1} = -\frac{x_1}{y_1}x + \frac{1}{y_1}$ and by $y = -\frac{b^2 x_2}{a^2 y_2}x + y_2 + \frac{b^2 x_2^2}{a^2 y_2} = -\frac{b^2 x_2}{a^2 y_2}x + \frac{b^2}{y_2}$; Hence the following system of equations holds:

$$-\frac{x_1}{y_1} = -\frac{b^2 x_2}{a^2 y_2} = m \quad (2)$$

$$\frac{1}{y_1} = \frac{b^2}{y_2} = B. \quad (3)$$

Using (2) we have $m^2 = \frac{x_1^2}{y_1^2} = \frac{x_1^2}{1-x_1^2}$, and $m^2 = \frac{b^4 x_2^2}{a^4 y_2^2} = \frac{b^4}{a^4} \frac{x_2^2}{\frac{b^2}{a^2}(a^2-x_2^2)} = \frac{b^2}{a^2} \frac{x_2^2}{a^2-x_2^2}$, which implies that $x_2^2 = \frac{a^4 m^2}{a^2 m^2 + b^2} = \frac{a^4 \frac{x_1^2}{1-x_1^2}}{a^2 \frac{x_1^2}{1-x_1^2} + b^2}$, which implies that

$$x_2^2 = \frac{a^4 x_1^2}{(a^2 - b^2)x_1^2 + b^2}. \quad (4)$$

Now (2) also implies that $\frac{x_1}{y_1} = \frac{b^2 x_2}{a^2 y_2}$, and (3) yields $\frac{x_1}{y_1} = x_1 \left(\frac{b^2}{y_2} \right)$, and hence $x_1 \left(\frac{b^2}{y_2} \right) = \frac{b^2 x_2}{a^2 y_2}$, which implies that $\frac{x_2}{x_1} = a^2$; (3) also yields $\frac{y_2}{y_1} = b^2$; Hence x_1 and x_2 must have the same sign, and y_1 and y_2 also must have the same sign; $\frac{x_2^2}{x_1^2} = a^4$ and (4) gives $(a^2 - b^2)x_1^2 + b^2 = 1$, which implies that

$$x_1^2 = \frac{1 - b^2}{a^2 - b^2}. \quad (5)$$

(5) and $x_2^2 = a^4 x_1^2$ then implies that

$$x_2^2 = a^4 \left(\frac{1 - b^2}{a^2 - b^2} \right). \quad (6)$$

Now it follows easily that if $a < 1$ (and thus $b < 1$ since $b < a$), then E_2 is contained in E_1 , which would imply that E_1 and E_2 cannot each be inscribed in the same convex quadrilateral. Similarly, if $b > 1$ (and thus $a > 1$ since $a > b$), then E_1 is contained in E_2 , and again E_1 and E_2 could not each be inscribed in the same convex quadrilateral. Thus $a \geq 1$ and $b \geq 1$, and since we assumed above that $a \neq 1 \neq b$, we must have $b < 1 < a$; (5) then yields $x_1 = \pm \sqrt{\frac{1-b^2}{a^2-b^2}}$, and for each choice of \pm sign for x_1 , one has $y_1 = \pm \sqrt{1-x_1^2} = \pm \sqrt{\frac{a^2-1}{a^2-b^2}}$. That yields four distinct points $Q_j = (x_1, y_1) = \left(\pm \sqrt{\frac{1-b^2}{a^2-b^2}}, \pm \sqrt{\frac{a^2-1}{a^2-b^2}} \right)$, $j = 1, 2, 3, 4$; Define the following four lines $y = mx + B$, where $m = -\frac{x_1}{y_1}$ and $B = \frac{1}{y_1}$ for each choice of (x_1, y_1) above: L_1 : $y = -\sqrt{\frac{1-b^2}{a^2-1}}x - \sqrt{\frac{a^2-b^2}{a^2-1}}$, L_2 : $y = \sqrt{\frac{1-b^2}{a^2-1}}x + \sqrt{\frac{a^2-b^2}{a^2-1}}$, L_3 : $y = \sqrt{\frac{1-b^2}{a^2-1}}x - \sqrt{\frac{a^2-b^2}{a^2-1}}$, and L_4 : $y = -\sqrt{\frac{1-b^2}{a^2-1}}x + \sqrt{\frac{a^2-b^2}{a^2-1}}$; Then it follows immediately that L_1, L_2, L_3 , and L_4 are tangent to E_1 at the Q_j since $m = -\frac{x_1}{y_1}$ and $B = \frac{1}{y_1}$; (6) then yields $x_2 = \pm a^2 \sqrt{\frac{1-b^2}{a^2-b^2}}$, where the $+$ or $-$ sign are chosen so that x_1 and x_2 have the same sign. Then $y_2 = \pm \frac{b}{a} \sqrt{a^2 - x_2^2} =$

$\pm \frac{b}{a} \sqrt{\frac{a^2 b^2 (a^2 - 1)}{a^2 - b^2}} = \pm b^2 \sqrt{\frac{a^2 - 1}{a^2 - b^2}}$, where again the $+$ or $-$ sign are chosen so that y_1 and y_2 have the same sign. Since $\frac{x_1}{y_1} = \frac{b^2 x_2}{a^2 y_2}$ and $\frac{1}{y_1} = \frac{b^2}{y_2}$, it follows that $m = -\frac{b^2 x_2}{a^2 y_2}$ and $B = \frac{b^2}{y_2}$, which implies that L_1, L_2, L_3 , and L_4 are also tangent to E_2 at the four distinct points $R_j = \left(\pm a^2 \sqrt{\frac{1-b^2}{a^2-b^2}}, \pm b^2 \sqrt{\frac{a^2-1}{a^2-b^2}} \right), j = 1, 2, 3, 4$; Now $L_1 \parallel L_4$ and $L_2 \parallel L_3$, which implies that L_1, L_2, L_3 , and L_4 must form a parallelogram. Furthermore, L_1, L_2, L_3 , and L_4 are the only lines which are common tangents to E_1 and E_2 since we have shown that $x_1 = \pm \sqrt{\frac{1-b^2}{a^2-b^2}}$ and $x_2 = \pm a^2 \sqrt{\frac{1-b^2}{a^2-b^2}}$ are the only solutions of (2) and (3). Thus E_1 and E_2 are each inscribed in the same convex quadrilateral, Q , and Q must be a parallelogram.

Now, to prove Theorem 2, one can use any nonsingular affine transformation to map Q to a quadrilateral of a simpler form. However, this will not work to prove Theorem 3 since the ratio of the eccentricity of two ellipses is **not** preserved in general under nonsingular affine transformations of the plane. For brevity, we want to use the same quadrilateral for the rest of this paper. So, by using an **isometry** of the plane, we can assume that Q has vertices $(0, 0), (0, u), (s, t)$, and (v, w) , where ■

$$s > 0, v > 0, u > 0, t > w. \quad (7)$$

The sides of Q , going clockwise, are given by $S_1 = \overline{(0, 0) (v, w)}$, $S_2 = \overline{(0, 0) (0, u)}$, $S_3 = \overline{(0, u) (s, t)}$, and $S_4 = \overline{(s, t) (v, w)}$; Let $L_1: y = \frac{w}{v}x$, $L_2: x = 0$, $L_3: y = u + \frac{t-u}{s}x$, and $L_4: y = w + \frac{t-w}{s-v}(x-v)$ denote the lines which make up the boundary of Q .

- Since Q is convex, (s, t) must lie above $\overrightarrow{(0, u) (v, w)}$ and (v, w) must lie below $\overrightarrow{(0, 0) (s, t)}$, which implies

$$v(t-u) + (u-w)s > 0, vt - ws > 0. \quad (8)$$

- Since no two sides of Q are parallel, $L_1 \nparallel L_3$ and $L_2 \nparallel L_4$, which implies

$$ws - v(t-u) \neq 0, s \neq v. \quad (9)$$

Assume now that (7), (8), and (9) hold throughout this section. Let

$$I = \begin{cases} (v/2, s/2) & \text{if } v < s \\ (s/2, v/2) & \text{if } s < v \end{cases}. \quad (10)$$

Note that

$$(s-2h)(2h-v) > 0, h \in I, \quad (11)$$

$$(s-2h)(s-v) > 0, h \in I, \quad (12)$$

$$(2h-v)(s-v) > 0, h \in I. \quad (13)$$

$M_1 = \left(\frac{1}{2}v, \frac{1}{2}(w+u)\right)$ and $M_2 = \left(\frac{1}{2}s, \frac{1}{2}t\right)$ are the midpoints of the diagonals of Q and the equation of the line thru M_1 and M_2 is

$$y = L(x) = \frac{t}{2} + \frac{w+u-t}{v-s} \left(x - \frac{s}{2}\right), x \in I. \quad (14)$$

The diagonals of Q are $D_1 = \overline{(0,0)(s,t)}$ = diagonal from lower left to upper right and $D_2 = \overline{(0,u)(v,w)}$ = diagonal from lower right to upper left.

Lemma 1 Define the following linear functions of h : $L_1(h) = 2(v(t-u) - ws)h + v(s(u+w) - vt)$, $L_2(h) = 2(v(u-t) + ws)h + s(v(t-2u) + s(u-w))$, $L_3(h) = (v(t-u) + (u-w)s)(s-2h)$, $L_4(h) = -2uh + vt + s(u-w)$, and $L_5(h) = 2(v(t-u) - ws)h + uvs$; Then $(s-v)L_j(h) > 0$ on I , $j = 1, 2, 3$, and $L_j(h) > 0$ on I , $j = 4, 5$.

Proof. Since each L_j is a linear function of h , it suffices to prove that $(s-v)L_j$ is non-negative at the endpoints of I , $j = 1, 2, 3$, and that L_j is non-negative at the endpoints of I , $j = 4, 5$. We have $L_1\left(\frac{v}{2}\right) = uv(s-v)$ and $L_1\left(\frac{s}{2}\right) = (s-v)(vt - ws)$, $L_2\left(\frac{v}{2}\right) = (s-v)(v(t-u) + (u-w)s)$ and $L_2\left(\frac{s}{2}\right) = su(s-v)$, and

$L_3\left(\frac{v}{2}\right) = (s-v)(v(t-u) + (u-w)s)$ and $L_3\left(\frac{s}{2}\right) = 0$; By (7) and (8), $(s-v)L_j(h) > 0$ on I , $j = 1, 2, 3$; Also,

$L_4\left(\frac{v}{2}\right) = v(t-u) + (u-w)s$ and $L_4\left(\frac{s}{2}\right) = vt - ws$, which implies that $L_4(h) > 0$ on I by (8). Finally, $L_5\left(\frac{v}{2}\right) = v(v(t-u) + (u-w)s)$ and

$L_5\left(\frac{s}{2}\right) = s(vt - ws)$, which implies that $L_5(h) > 0$ on I , by (7) and (8). ■

Now define the following cubic polynomial, where L_5 is given by Lemma 1.

$$R(h) = (s-2h)(2h-v)L_5(h). \quad (15)$$

Note that the three roots of R are $h_1 = \frac{1}{2}v$, $h_2 = \frac{1}{2}s$, and $h_3 = \frac{1}{2} \frac{uvs}{ws - v(t-u)}$;

By (11) and Lemma 1,

$$R(h) > 0 \text{ on } I. \quad (16)$$

We now state a result, without proof, about when a quadratic equation in x and y yields an ellipse. The first condition ensures that the conic is an ellipse, while the second condition ensures that the conic is nondegenerate.

Lemma 2 The equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, with $A, C > 0$, is the equation of an ellipse if and only if $4AC - B^2 > 0$ and $CD^2 + AE^2 - BDE - F(4AC - B^2) > 0$.

The following proposition gives the equation of an ellipse inscribed in the quadrilateral Q with vertices $(0,0)$, $(0,u)$, (v,w) , and (s,t) . We obtained (17)

below using some results from [3] along with a method for obtaining the equation of an ellipse given its foci and the lengths of the axes of the ellipse. We do not provide those somewhat cumbersome details here. This equation was not given in [3]. Rather, we state the equation below and prove that the equation works using elementary calculus. One could, of course, attempt to derive the equation of an ellipse inscribed in the quadrilateral Q with vertices $(0, 0)$, $(0, u)$, (v, w) , and (s, t) by solving a system of nonlinear equations for the unknown coefficients, A thru F , of the equation, and for the unknown points of tangency. We tried this for a simpler quadrilateral and it is somewhat messy.

Proposition 2 *Let Q be the quadrilateral with vertices $(0, 0)$, $(0, u)$, (v, w) , and (s, t) , which satisfy (7), (8), and (9). Let I be given by (10) and let L be given by (14). Then E_0 is an ellipse inscribed in Q if and only if the general equation of E_0 is given by*

$$\begin{aligned} & 4(s-v)((s-v)L^2(h) + uw(2h-s))(x-h)^2 \\ & + 4(s-v)^2 h^2 (y-L(h))^2 + 4(s-v) \times \\ & ((-2t+2u+2w)h^2 + \\ & (vt-su-ws-2vu)h + svu)(x-h)(y-L(h)) \\ & = uR(h)0, h \in I. \end{aligned} \quad (17)$$

Proof. Suppose that the general equation of E_0 is given by (17) for fixed $h \in I$; Then we can write the equation of E_0 as $\Psi(x, y) = 0$, where $\Psi(x, y) = A(h)x^2 + B(h)xy + C(h)y^2 + D(h)x + E(h)y + F(h)$, and

$$\begin{aligned} A(h) &= 4(s-v)^2 \left(\left(\frac{1}{2}t + \frac{w+u-t}{v-s} \left(h - \frac{1}{2}s \right) \right)^2 + \frac{uw(2h-s)}{s-v} \right), \\ B(h) &= 4(s-v)(2(u+w-t)h^2 + (v(t-2u) - s(u+w))h + uvs), \\ C(h) &= 4(s-v)^2 h^2, \\ D(h) &= 2u(2h-s)(2(v(w+t-u) - 2ws)h + v(s(u+w) - vt)), \\ E(h) &= 4uv(s-v)h(2h-s), \text{ and} \\ F(h) &= u^2v^2(2h-s)^2. \end{aligned} \quad (18)$$

First we want to show that $\Psi(x, y) = 0$ defines the equation of an ellipse. Substituting for $A(h)$ thru $F(h)$ using (18) and simplifying gives

$$\begin{aligned} & 4A(h)C(h) - B^2(h) = 16u(s-v)^2 R(h), \\ & (C(h)D^2(h) + A(h)E^2(h) - B(h)D(h)E(h)) - F(h)(4A(h)C(h) - B^2(h)) = \\ & = 16(s-v)^2 u^2 R^2(h). \end{aligned} \quad (19)$$

By (16) and Lemma 2, (17) defines the equation of an ellipse for any $h \in I$. Now let the L_j 's be given as in Lemma 1. Define the following points, which we will show shortly are the points of tangency of E_0 with Q :

- $\zeta_1 = \lambda_1(v, w) + (1 - \lambda_1)(0, 0)$, where $\lambda_1 = \frac{(s - 2h)uv}{L_1(h)}$, which implies that
 $1 - \lambda_1 = \frac{(vt - ws)(2h - v)}{L_1(h)}.$
- $\zeta_2 = \lambda_2(0, u) + (1 - \lambda_2)(0, 0)$, where $\lambda_2 = \frac{(s - 2h)v}{2h(s - v)}$, which implies that
 $1 - \lambda_2 = \frac{s(2h - v)}{2h(s - v)}.$
- $\zeta_3 = \lambda_3(s, t) + (1 - \lambda_3)(0, u)$, where $\lambda_3 = \frac{(2h - v)su}{L_2(h)}$, which implies that
 $1 - \lambda_3 = \frac{(v(t - u) + (u - w)s)(s - 2h)}{L_2(h)}.$
- $\zeta_4 = \lambda_4(v, w) + (1 - \lambda_4)(s, t)$, where $\lambda_4 = \frac{L_3(h)}{(s - v)L_4(h)}$, which implies
that $1 - \lambda_4 = \frac{(vt - ws)(2h - v)}{(s - v)L_4(h)}.$

Then $0 < \zeta_j < 1, j = 1, 2, 3, 4$ by Lemma 1 and (11)–(13); Hence $\zeta_j \in S_j, j = 1, 2, 3, 4$, where S_j denote the sides of Q going clockwise with $S_1 = (0, 0)(v, w)$; It follows easily that $\Psi(\zeta_j) = 0, j = 1, 2, 3, 4$, which implies that $\zeta_j \in E_0, j = 1, 2, 3, 4$; For fixed $h \in I$, let $Z(x, y) = -\frac{\partial \Psi / \partial x}{\partial \Psi / \partial y}$, which represents the slope, $\frac{dy}{dx}$, of the ellipse. Then $Z(\zeta_1) = \frac{w}{v} = \text{slope of } L_1, Z(\zeta_3) = \frac{t - u}{s} = \text{slope of } L_3$, and $Z(\zeta_4) = \frac{t - w}{s - v} = \text{slope of } L_4$; Hence E_0 is tangent to S_1, S_3 , and S_4 at the points ζ_1, ζ_3 , and ζ_4 respectively. Also, $\frac{\partial \Psi}{\partial y}(\zeta_2) = 0$ and $\frac{\partial \Psi}{\partial x}(\zeta_2) = -\frac{2u}{h}(2h - v)(s - 2h)L_5(h) \neq 0$ by Lemma 1 and (11), which implies that E_0 is tangent to the vertical line segment S_2 at ζ_2 . For any simple closed convex curve, such as an ellipse, tangent to each side of Q then implies that that curve lies in Q . That proves that E_0 is inscribed in Q . Second, suppose that E_0 is an ellipse inscribed in Q . By Theorem 1, E_0 has center $(h_1, L(h_1))$ for some $h_1 \in I$. We have just shown that (17) represents a family of ellipses inscribed in Q as h varies over I , and it is not hard to show that each ellipse given by (17) has center $(h, L(h))$ for some $h \in I$. Let \tilde{E}_0 be the ellipse given by (17) with $h = h_1$. Hence \tilde{E}_0 also has center $(h_1, L(h_1))$ and is inscribed in Q . Since $(0, 0), (0, u), (v, w)$, and (s, t) satisfy (9), Q is not a parallelogram. Then by Proposition 1, $\tilde{E}_0 = E_0$ and the general equation of E_0 must be given by (17).

■

Proof. (proof of Theorem 2): *Existence* follows immediately from Proposition 2. We already proved *uniqueness* above in the proof of Proposition 2. ■

3 Preliminary Lemmas

Lemma 3 Let $G(h) = \frac{J(h) - \sqrt{M(h)}}{J(h) + \sqrt{M(h)}}$, where J and M are differentiable functions on some interval, I , with $J(h) + \sqrt{M(h)} \neq 0$ on I and $M(h) > 0$ on I . Then G is differentiable on I and $G'(h) = \frac{2J'(h)M(h) - J(h)M'(h)}{\sqrt{M(h)}(J(h) + \sqrt{M(h)})^2}$ for $h \in I$.

Proof. It is clear that G is differentiable on I , and the rest of the lemma follows from the quotient rule after some simplification. ■

Before proving Theorem 3, we state and prove the following series of lemmas. The purpose of these lemmas will be to show that the eccentricity of an inscribed ellipse, as a function of h , has a unique root in I , where I is given by (10) throughout this section. We also want to find a formula for that root as well. Assume throughout that (7), (8), and (9) hold.

First we define the following quadratic polynomial in h :

$$\begin{aligned} o(h) &= -2(s^2 + t^2)(s - v)h^2 - 2Kh + sK, \\ K &= (s^2 + t^2)v^2 - 2ws(vt - ws). \end{aligned} \quad (20)$$

Lemma 4 : $K > 0$.

Proof. Writing K from (20) as a quadratic in v , $K(v) = (s^2 + t^2)v^2 - 2wstv + 2s^2w^2$, the discriminant of $K(v)$ is $(-2wst)^2 - 4(s^2 + t^2)(2s^2w^2) = -4s^2w^2(t^2 + 2s^2) < 0$, which implies that $K(v)$ has no real roots. Since $K(0) > 0$, it follows that $K(v) > 0$, for all real v . ■

Lemma 5 Let $p_1 = v^2(s^2 + t^2) - 4ws(vt - ws)$; Then $p_1 > 0$.

Proof. Rewrite $p_1 = (2ws - vt)^2 + v^2s^2 > 0$ since $s, v \neq 0$. ■

Lemma 6 o has exactly one root in I .

Proof.

$$o\left(\frac{v}{2}\right) = \frac{1}{2}(s - v)p_1, o\left(\frac{s}{2}\right) = -\frac{1}{2}s^2(s - v)(s^2 + t^2). \quad (21)$$

By Lemma 5, $o\left(\frac{v}{2}\right)o\left(\frac{s}{2}\right) < 0$, which implies that o has an odd number of roots in I ; Since o is a quadratic, o must have one root in I . Note that since o has two distinct real roots, the discriminant of o , $4K^2 + 8(s^2 + t^2)(s - v)sK = 4K(2(s^2 + t^2)s(s - v) + K)$, is positive. By Lemma 4, $2(s^2 + t^2)s(s - v) + K > 0$ and by the quadratic formula, the roots of o are ■

$$\begin{aligned} h_- &= \sqrt{K} \frac{-\sqrt{K} - \sqrt{2(s^2 + t^2)s(s - v) + K}}{2(s^2 + t^2)(s - v)}, \\ h_+ &= \sqrt{K} \frac{-\sqrt{K} + \sqrt{2(s^2 + t^2)s(s - v) + K}}{2(s^2 + t^2)(s - v)}. \end{aligned} \quad (22)$$

Note that $h_+ - h_- = \frac{\sqrt{K}\sqrt{2(s^2+t^2)s(s-v)+K}}{(s^2+t^2)(s-v)}$, which implies that

$$\begin{cases} h_- > h_+ & \text{if } s < v \\ h_+ > h_- & \text{if } v < s \end{cases}.$$

The following lemma allows us to find the unique root of o in I .

Lemma 7 h_+ is the unique root of o in I .

Proof. Case 1: $s > v$; Then $I = \left(\frac{v}{2}, \frac{s}{2}\right)$ and $h_- < 0$, which implies that $h_- \notin I$; By Lemma 6, $h_+ \in I$.

Case 2: $s < v$; Then $I = \left(\frac{s}{2}, \frac{v}{2}\right)$; Since $\lim_{h \rightarrow \infty} o(h) = \infty$ and $o\left(\frac{v}{2}\right) < 0$ by (22) and Lemma 5, o has a root in the interval $\left(\frac{v}{2}, \infty\right)$; Hence the smaller of the two roots of o must lie in I ; Since $h_+ < h_-$, one must have $h_+ \in I$. ■

Now define the polynomials

$$\begin{aligned} J(h) &= A(h) + C(h), \\ M(h) &= (A(h) - C(h))^2 + (B(h))^2, \end{aligned} \quad (23)$$

where $A(h), B(h)$, and $C(h)$ are given by (18). Note that $M(h) \geq 0$; Some simplification yields

$$J^2(h) - M(h) = 16u(s-v)^2 R(h), \quad (24)$$

where R is given by (15).

Lemma 8 $J(h) > 0, h \in I$.

Proof. If $J(h_0) = 0$ for some $h_0 \in I$, then $R(h_0) \leq 0$ by (24), which contradicts (16); Since $J\left(\frac{v}{2}\right) = (s-v)^2((w-u)^2+v^2) > 0$, it follows that $J(h) > 0, h \in I$. ■

Lemma 9 Suppose that Q has vertices $(0,0), (0,u), (s,t)$, and (v,w) , where s, t, u, v, w satisfy (7), (8), and (9).

(i) Q is a type 1 midpoint diagonal quadrilateral if and only if

$$u = \frac{vt - ws}{s}. \quad (25)$$

(ii) Q is a type 2 midpoint diagonal quadrilateral if and only if

$$u = \frac{vt - ws}{2v - s}. \quad (26)$$

Proof. Recall that $L(x) = \frac{t}{2} + \frac{w+u-t}{v-s}\left(x - \frac{s}{2}\right)$. The line containing D_1 is $y = \frac{t}{s}x$; Then $D_1 = L \iff \frac{w+u-t}{v-s} = \frac{t}{s}$ and $\frac{t}{2} - \frac{s}{2} \frac{w+u-t}{v-s} =$

$0 \iff (25)$ holds. That proves (i). The line containing D_2 is $y = u + \frac{w-u}{v}x$; Then $D_2 = L \iff \frac{w+u-t}{v-s} = \frac{w-u}{v}$ and $\frac{t}{2} - \frac{s}{2} \frac{w+u-t}{v-s} = u \iff (2v-s)u + ws - vt = 0$; Note that if $2v-s=0$, then $(2v-s)u + ws - vt \neq 0$ since $ws - vt \neq 0$ by (8); Thus Q is a type 2 midpoint diagonal quadrilateral if and only if (26) holds. That proves (ii). ■

We now prove two results about midpoint diagonal quadrilaterals. We define a trapezoid to be a quadrilateral with at least one pair of parallel sides.

Lemma 10 *Suppose that Q is a midpoint diagonal quadrilateral which is also a trapezoid. Then Q is a parallelogram.*

Proof. Suppose that Q is a midpoint diagonal quadrilateral which is a trapezoid, but which is not a parallelogram. By affine invariance, we may assume that Q is the trapezoid with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,t)$, $0 < t \neq 1$; The diagonals of Q are $D_1: y = tx$ and $D_2: y = 1-x$, and the midpoints of the diagonals are $M_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $M_2 = \left(\frac{1}{2}, \frac{1}{2}t\right)$; The open line segment, L , joining M_1 and M_2 is contained in the vertical line $x = \frac{1}{2}$; Since the diagonals of Q are nonvertical lines, $D_1 \neq L$ and $D_2 \neq L$, which implies that Q is not a midpoint diagonal quadrilateral. ■

Let D_1 denote the diagonal of Q from lower left to upper right and let D_2 denote the diagonal of Q from lower right to upper left. We note here that there are two types of midpoint diagonal quadrilaterals: Type 1, where $D_1 = L$ and Type 2, where $D_2 = L$.

Lemma 11 *Suppose that Q is both a tangential and a midpoint diagonal quadrilateral. Then Q is an orthodiagonal quadrilateral.*

Proof. Since an isometry preserves both tangential and midpoint diagonal quadrilaterals (a general affine transformation does not suffice), we can assume that Q has vertices $(0,0)$, $(0,u)$, (s,t) , and (v,w) , where s, t, u, v, w satisfy (7), (8), and (9). Now Q is tangential $\iff Z = 0$, where

$$Z = \left((v^2 + w^2)(s^2 + (t-u)^2) - (tu - vs - wt)^2 - u^2((s-v)^2 + (t-w)^2) \right)^2 - 4(u(tu - vs - wt))^2((s-v)^2 + (t-w)^2). \quad (27)$$

Case 1: Q is a type 1 midpoint diagonal quadrilateral. Substituting for u in (27) using (25) yields

$$Z = \frac{-4(s-v)^2(vt - ws)^2(v(t^2 - s^2) - 2wst)^2}{s^4} = 0$$

\iff

$$v(t^2 - s^2) - 2wst = 0. \quad (28)$$

So if Q is both a tangential and a type 1 midpoint diagonal quadrilateral, then (28) holds. The slopes of the diagonals are $m_1 = \frac{t}{s}$ and $m_2 = \frac{w-u}{v} = \frac{2ws-vt}{vs}$, which implies that $m_1 m_2 + 1 = \frac{t}{s} \frac{2ws-vt}{vs} + 1 = \frac{2wts - v(t^2 - s^2)}{s^2 v} = 0$ by (28).

Case 2: Q is a type 2 midpoint diagonal quadrilateral. Substituting for u in (27) using (26) yields

$$Z = \frac{-4v^2(s-v)^2(vt-ws)^2(2(vs+wt)-(s^2+t^2))^2}{(2v-s)^4} = 0$$

if and only if

$$2(vs+wt)-(s^2+t^2) = 0. \quad (29)$$

So if Q is both a tangential and a type 2 midpoint diagonal quadrilateral, then (29) holds. The slopes of the diagonals are $m_1 = \frac{t}{s}$ and $m_2 = \frac{w-u}{v} = \frac{2w-t}{2v-s}$, which implies that $m_1 m_2 + 1 = \frac{t}{s} \frac{2w-t}{2v-s} + 1 = \frac{2(vs+wt)-(s^2+t^2)}{s(2v-s)} = 0$ by (29). ■

4 Main Result

The following lemma allows us to express the eccentricity of an ellipse as a function of the coefficients of an equation of that ellipse.

Lemma 12 *Suppose that E_0 is an ellipse with equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$; Let a and b denote the lengths of the semi-major and semi-minor axes, respectively, of E_0 . Then*

$$\frac{b^2}{a^2} = \frac{A+C-\sqrt{(A-C)^2+B^2}}{A+C+\sqrt{(A-C)^2+B^2}}. \quad (30)$$

Proof. By [5],

$$\begin{aligned} a^2 &= \delta \frac{A+C+\sqrt{(A-C)^2+B^2}}{2} \\ b^2 &= \delta \frac{A+C-\sqrt{(A-C)^2+B^2}}{2}, \end{aligned} \quad (31)$$

where $\delta = 4 \frac{(CD^2 + AE^2 - BDE) - F(4AC - B^2)}{(4AC - B^2)^2}$; Note that $\delta > 0$ by

Lemma 2. (30) then follows immediately from (31). ■

We now state and prove our main result, which gives a geometric characterization of the unique ellipse of minimal eccentricity inscribed in a midpoint diagonal quadrilateral.

Theorem 3 (i) *There is a unique ellipse of minimal eccentricity, E_I , inscribed in a midpoint diagonal quadrilateral, Q .*

(ii) *Furthermore, the smallest non-negative angle between equal conjugate diameters of E_I equals the smallest non-negative angle between the diagonals of Q .*

Remark 3 *In [3] we proved that there is a unique ellipse of minimal eccentricity inscribed in any convex quadrilateral, Q ; The uniqueness for midpoint diagonal quadrilaterals would then follow from that result. However, the proof here, specialized for midpoint diagonal quadrilaterals, is self-contained, uses different methods, and does not require the result from [3].*

Proof. If Q is a parallelogram, then Theorem 3 was proven in [4]. Now suppose that Q is **not** a parallelogram. Then by Lemma 10, Q is **not** a trapezoid. Thus by using an isometry of the plane, we may assume that Q has vertices $(0, 0)$, $(0, u)$, (s, t) , and (v, w) , where s, t, u, v , and w satisfy (7), (8), and (9). If E_0 is an ellipse inscribed in Q , then by Proposition 2, the equation of E_0 is $A(h)x^2 + B(h)xy + C(h)y^2 + D(h)x + E(h)y + F(h) = 0$ for some $h \in I$, where $A(h)$ thru $F(h)$ are given by (18). Let $a = a(h)$ and $b = b(h)$ denote the lengths of the semi-major and semi-minor axes, respectively, of E_0 . Since the square of the eccentricity of E_0 equals $1 - \frac{b^2}{a^2}$, it suffices to maximize $\frac{b^2}{a^2}$, which is really a function of $h \in I$ since we allow E_0 to vary over all ellipses inscribed in Q ; Thus we want to maximize $G(h), h \in I$, where $G(h) = \frac{b^2(h)}{a^2(h)}$; Using (30),

$$G(h) = \frac{A(h) + C(h) - \sqrt{(A(h) - C(h))^2 + (B(h))^2}}{A(h) + C(h) + \sqrt{(A(h) - C(h))^2 + (B(h))^2}} = \frac{J(h) - \sqrt{M(h)}}{J(h) + \sqrt{M(h)}}, \text{ where}$$

J and M are given by (23). By Lemma 3,

$$G'(h) = \frac{p(h)}{\sqrt{M(h)}(J(h) + \sqrt{M(h)})^2}, \quad (32)$$

where p is the quartic polynomial given by

$$p(h) = 2J'(h)M(h) - J(h)M'(h). \quad (33)$$

We now prove Theorem 3 for the case when Q is a type 1 midpoint diagonal quadrilaterals—the proof for Type 2 midpoint diagonal quadrilaterals being similar. So we now adapt the formulas in (18) and in (30) to type 1 midpoint diagonal quadrilaterals.

Uniqueness: First we show that there is a unique ellipse of minimal eccentricity, E_I , inscribed in Q . Then we shall prove the property about the angle between equal conjugate diameters of E_I . As earlier, let L be the line thru the midpoints of the diagonals of Q , so that the equation of L is given by (14); Let $m_1 = \frac{t}{s}$ and $m_2 = \frac{w-u}{v}$ denote the slopes of D_1 and D_2 , respectively; By

Lemma 9, Q is Type 1 if and only if (25) holds. Substituting for u in (33) using (25) yields $p(h) = 256h \left(\frac{s-v}{s} \right)^4 (vt - ws)^2 (s-h) o(h)$, where o is given by (20). Note that Lemma 8 implies that

$$J(h) + \sqrt{M(h)} > 0, h \in I. \quad (34)$$

Now assume first that Q is a tangential quadrilateral. Then Q is an orthodiagonal quadrilateral by Lemma 11, and so the diagonals of Q are perpendicular. Also, there is a unique circle, Φ , inscribed in Q , which implies that Φ is the unique ellipse of minimal eccentricity inscribed in Q since Φ has eccentricity 0. Since any pair of perpendicular diameters of a circle are equal conjugate diameters, the smallest non-negative angle between conjugate diameters of a circle is $\frac{\pi}{2}$. Hence Theorem 3 holds when Q is a tangential quadrilateral. So assume now that Q is **not** a tangential quadrilateral, which implies that $A(h) \neq C(h)$ for all $h \in I$ and thus $M(h) \neq 0$ for all $h \in I$ by (23); Since M is non-negative we have

$$M(h) > 0, h \in I. \quad (35)$$

By (32), (34), and (35), G is differentiable on I . By (32) and (33),

$$G'(h) = \frac{256h \left(\frac{s-v}{s} \right)^4 (vt - ws)^2 (s-h) o(h)}{\sqrt{M(h)}(J(h) + \sqrt{M(h)})^2}. \quad (36)$$

By Lemma 7 and (36), h_+ is the unique root of G' in I , where h_+ is given by (22). Since $G(h) = \frac{b^2(h)}{a^2(h)}$, it follows that $G(h) > 0$ on I ; Also, $G\left(\frac{v}{2}\right) = G\left(\frac{s}{2}\right) = 0$; Since G is positive in the interior of I and vanishes at the endpoints of I , h_+ must yield the global maximum of G on I . That proves uniqueness.

Angle between equal conjugate diameters: We now prove that the smallest non-negative angle between equal conjugate diameters of E_I equals the smallest non-negative angle between the diagonals of Q . First we find a simplified formula for $G(h_+)$; Solving for h_+^2 in the equation $o(h_+) = 0$ yields

$$h_+^2 = \frac{(s - 2h_0)K}{2(s^2 + t^2)(s - v)}, \quad (37)$$

where K is given by (20). Substituting for u again in the formulas for $A(h)$ and $B(h)$ from (18) using (25) and simplifying gives

$$A(h_+) + C(h_+) = 4(s - v) \times \left(\frac{(s - v)(t^2 + s^2)}{s^2} h_+^2 + \frac{2w(vt - ws)}{s} h_+ - w(vt - ws) \right) \quad (38)$$

and

$$A(h_+) - C(h_+) = 4(s-v) \times \left(\frac{K(t^2 - s^2)(s - 2h_+)}{2s^2(s^2 + t^2)} + \frac{2w(vt - ws)}{s} h_+ - w(vt - ws) \right). \quad (39)$$

Using (38) and (37) and simplifying then gives

$$J(h_+) = \frac{2p_1(s-v)}{s^2}(s - 2h_+). \quad (40)$$

(37) also yields

$$B(h_+) = -\frac{4(s-v)w(2wst - (t^2 - s^2)v)}{s^2 + t^2}(s - 2h_+). \quad (41)$$

Using (41), (37), and (39) gives

$$M(h_+) = \frac{4p_1(s-v)^2}{(s^2 + t^2)s^4} \times (s - 2h_+)^2(2wst - (t^2 - s^2)v)^2, \quad (42)$$

where M is given by (23). Note that $2wst - (t^2 - s^2)v \neq 0$ since $M(h_+) > 0$ by (35); Also, $h_+ \in I$ implies that $(s-v)(2h_+ - s) < 0$; Thus (42) yields

$$\sqrt{M(h_+)} = \frac{2\sqrt{p_1}(s-v)|2wst - (t^2 - s^2)v|(s - 2h_+)}{s^2\sqrt{s^2 + t^2}}. \quad (43)$$

By (40) and (43) we have

$$\begin{aligned} \left(J(h_+) + \sqrt{M(h_+)} \right)^2 &= \frac{4p_1(s-v)^2(s - 2h_+)^2}{s^4} \times \\ &\left(\sqrt{p_1} + \frac{|2wst - (t^2 - s^2)v|}{\sqrt{s^2 + t^2}} \right)^2. \end{aligned} \quad (44)$$

(40) and (42) imply that

$$\begin{aligned} J^2(h_+) - M(h_+) &= \frac{4p_1(s-v)^2(s - 2h_+)^2}{s^4} \times \\ &\left(p_1 - \frac{(2wst - (t^2 - s^2)v)^2}{s^2 + t^2} \right). \end{aligned} \quad (45)$$

$$\text{By (44) and (45) we have } G(h_+) = \frac{J^2(h_+) - M(h_+)}{(J(h_+) + \sqrt{M(h_+)})^2} =$$

$$\frac{p_1 - \frac{(2wst - (t^2 - s^2)v)^2}{s^2 + t^2}}{\left(\sqrt{p_1} + \frac{|2wst - (t^2 - s^2)v|}{\sqrt{s^2 + t^2}}\right)^2}, \text{ which implies that}$$

$$G(h_+) = \frac{\sqrt{s^2 + t^2}\sqrt{p_1} - |2wst - (t^2 - s^2)v|}{\sqrt{s^2 + t^2}\sqrt{p_1} + |2wst - (t^2 - s^2)v|}. \quad (46)$$

Now let α denote the smallest non-negative angle between the diagonals of Q , so that $0 \leq \alpha \leq \frac{\pi}{2}$, and let $m_1 = \frac{t}{s}$ and $m_2 = \frac{w-u}{v}$ denote the slopes of the diagonals of Q ; Substituting for u using (25) yields $\frac{m_2 - m_1}{1 + m_1 m_2} = \frac{2s(vt - ws)}{(t^2 - s^2)v - 2wts}$; Using the formula $\tan \alpha = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|$ then yields, by (8),

$$\tan \alpha = \frac{2s(vt - ws)}{|(t^2 - s^2)v - 2wts|}. \quad (47)$$

Now let τ_1 and τ_2 denote a pair of *equal* conjugate diameters of any ellipse, E_0 ; Let θ denote the acute angle going counterclockwise from the major axis of E_0 to one of the equal conjugate diameters, and let $\Gamma =$ angle between the *equal* conjugate diameters of E_0 , $0 \leq \Gamma \leq \pi$; If a and b are the lengths of the semi-major and semi-minor axes, respectively, of E_0 , then it is known that τ_1 and τ_2 make *equal acute angles*, on *opposite* sides, with the major axis of E_0 ; Thus $\Gamma = 2\theta$ and $\tan \theta = \frac{b}{a}$ (see page 170 of [6]). Note that Salmon refers to θ as the angle τ_1 or τ_2 makes with the axis of x ; But Salmon is assuming the ellipse has major axis = x axis; So θ is really the angle with the major axis of E_0 ; Assume now for the rest of the proof that $E_0 = E_I$, the unique ellipse of minimal eccentricity inscribed in Q ; We then want to show that $\alpha = \Gamma$, which is equivalent to showing that $\tan 2\theta = \tan \alpha$, which in turn is equivalent to showing that $\tan^2 2\theta = \tan^2 \alpha$ since we are assuming that θ and α lie in the first quadrant. Now $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2 \frac{b}{a}}{1 - \frac{b^2}{a^2}}$, which implies that

$$\tan^2 2\theta = \frac{4 \left(\frac{b^2}{a^2} \right)}{\left(1 - \frac{b^2}{a^2} \right)^2}; \text{ As shown above in the first part of the proof, } h_+ \text{ must}$$

yield the global maximum of G on I . $G(h_+) = \frac{b^2}{a^2}$ implies that

$$\tan^2 2\theta = \frac{4G(h_+)}{(1 - G(h_+))^2}. \quad (48)$$

Thus we must show that

$$\frac{4G(h_+)}{(1 - G(h_+))^2} = \left(\frac{2s(vt - ws)}{(t^2 - s^2)v - 2wts} \right)^2. \quad (49)$$

Using (46), $1 - G(h_+) = \frac{2|2wst - (t^2 - s^2)v|}{\sqrt{s^2 + t^2}\sqrt{p_1} + |2wst - (t^2 - s^2)v|}$, which implies that

$$(1 - G(h_+))^2 = \frac{4(2wst - (t^2 - s^2)v)^2}{(\sqrt{s^2 + t^2}\sqrt{p_1} + |2wst - (t^2 - s^2)v|)^2}. \quad (50)$$

(49) then follows from (46) and (50). ■

Remark 4 We do not know if it is possible for the second part of Theorem 3 to hold when Q is not a midpoint diagonal quadrilateral. If there were a quadrilateral, Q , such that Q is both a tangential and an orthodiagonal quadrilateral, but not a midpoint diagonal quadrilateral, then it would follow easily that the second part of Theorem 3 holds. However, it is not hard to show that if Q is both a tangential and an orthodiagonal quadrilateral, then Q must be a midpoint diagonal quadrilateral.

Remark 5 Suppose that Q is a Type 2 midpoint diagonal quadrilateral. If one reflects Q thru the y axis one obtains a Type 1 midpoint diagonal quadrilateral. Thus it might appear that one need only prove Theorem 3 for Type 1 midpoint diagonal quadrilaterals. However, in the proof above we also assumed that Q has vertices $(0, 0)$, $(0, u)$, (s, t) , and (v, w) . Reflection thru the y axis does not preserve the form of those vertices.

5 Example

Suppose that Q has vertices $(0, 0)$, $(0, u)$, (s, t) , and (v, w) , where $s = 4$, $t = 6$, $v = 2$, $w = 1$, and $u = 2$; Then s, t, u, v, w satisfy (7), (8), and (9), and $I = (1, 2)$; Q is a type 1 midpoint diagonal quadrilateral since $u = \frac{vt - ws}{s}$;

$\tan \mu = \frac{2s(vt - ws)}{|(t^2 - s^2)v - 2wts|} = 8$; $o(h) = 16(-13h^2 - 18h + 36)$, which has roots

$$\frac{3}{13}(-3 \pm \sqrt{61}); \quad h_+ = \frac{3}{13}(-3 + \sqrt{61}) \quad \text{and} \quad \frac{4G(h_+)}{(1 - G(h_+))^2} = \frac{4\left(\frac{33 - \sqrt{65}}{32}\right)}{\left(1 - \frac{33 - \sqrt{65}}{32}\right)^2} =$$

$64 = \tan^2 \mu$; The equation of E_I is

$$\begin{aligned} & (35 - 3\sqrt{61})(29x^2 - 4xy + 36y^2) \\ & + 48(72 - 11\sqrt{61})(x + 2y) \\ & + 16(887 - 105\sqrt{61}) = 0. \end{aligned}$$

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